

Continuum Limits for Critical Percolation and Other Stochastic Geometric Models

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Abstract

The talk presented at ICMP 97 focused on the scaling limits of critical percolation models, and some other systems whose salient features can be described by collections of random lines. In the scaling limit we keep track of features seen on the macroscopic scale, in situations where the short-distance scale at which the system's basic variables are defined is taken to zero. Among the challenging questions are the construction of the limit, and the explanation of some of the emergent properties, in particular the behavior under conformal maps as discussed in ref. [1]. A descriptive account of the project, and some related open problems, is found in ref. [2] and in [3] (joint work with A. Burchard) where tools are developed for establishing a curve-regularity condition which plays a key role in the construction of the limit. The formulation of the scaling limit as a random Web measure permits to formulate the question of uniqueness of measure(s) describing systems of random curves satisfying the conditions of independence, Euclidean invariance, and regularity. The *uniqueness* question remains open; progress on it could shed light on the purported universality of critical behavior and the apparent conformal invariance of the critical measures. The random Web yields also another perspective on some of the equations of conformal field theory which have appeared in this context, such as the equation proposed by J. Cardy [4].

1 Comments on Selected Points

The connected clusters, conveniently viewed as the conducting clusters, in *critical* percolation models are well known to assume “fractal” characteristics [5], extending tenuously over many scales. Our goal is to formulate a macroscopic description of such systems in terms which remains meaningful in the scaling limit, where the scale at which the model is constructed drops out of sight (is taken to zero). The interest in this question is enhanced by the observation that the correlations in various spin models have a geometric representation in terms of random cluster with somewhat similar characteristics [6, 7], which suggests that the resulting system of random collections of lines may bear interesting relations to certain random fields. There is accumulating evidence for conformal invariance of the critical models [1, 4, 8, 9], and specific equations were surmised on the basis of analogies with random fields. It would be interesting to understand such equations in terms which are native to the microscopic model.

Following are few words to indicate some of the relevant points. Further discussion, and a better reference list is provided in Ref. [2], from which also the figures are taken, and Ref. [3].

- There are differences between the **macroscopic** and the **microscopic** perspectives on a physical system. While the difference is not hard to understand, ignoring it one runs the peril of paradoxes (ranging from Zeno's puzzles up to some more recent lively discussions concerning the non-uniqueness of the Incipient Spanning Cluster(s) [10].

The microscopic view has been mathematically formulated, and extensively studied, within the infinite-system formalism. New constructs are required for the macroscopic view. Field theory

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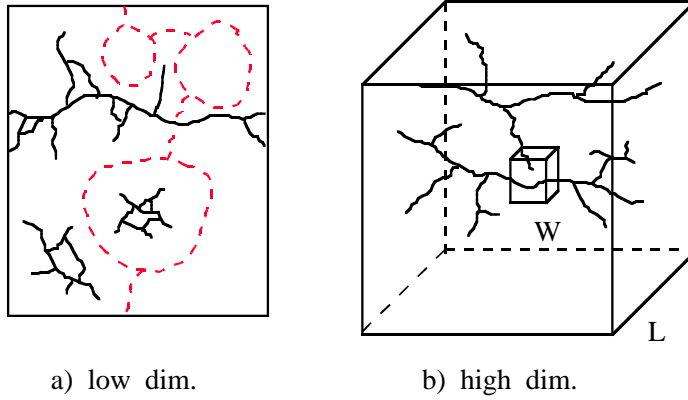


Figure 1: (After ref. [2]) Schematic depiction of the macroscopic view of the connected clusters in critical percolation models. a) In $d = 2$, and presumably other “low dimensions” on each scale one would typically find a finite number of distinct clusters within a given compact region. The obstructions are likewise present. b) In high dimensions (presumably all $d > 6$) the clusters proliferate and extend: the probability that there are at least k spanning cluster tends to one for each $k < \infty$, even if one considers only clusters touching a preassigned region whose size shrinks as $W \approx \delta^{|d-6|+/(d-4)-o(1)}$ or alternatively cluster reaching exceeding distances away.

provides the relevant tool for systems with order parameter. However, there is room and need for other tools to capture the stochastic geometric features seen, e.g., in percolation models.

◇ The seemingly natural formulation, in which the geometric features are expressed through the random clusters, viewed as closed subsets of the Euclidean space with the notion of distance based on the Hausdorff metric, is inadequate. The problem is caused by the ubiquity of choke points: typical configurations exhibit many sites at which a microscopic change (often just the flip of a single bond) produces a macroscopic scale effect.

• It is proposed to base the macroscopic description of such models on the collection, henceforth called the *Web*, of all the realized (connecting) curves, which are self-avoiding in the suitable sense and supported on the connected clusters. Alternatively, though this notion is less developed at present, one may describe the configuration in terms of the connected clusters but change the notion of distance from the Hausdorff metric to one based on the connected curves they support. For convenience, in 2D one may simultaneously keep track of the dual system of obstructions, which forms another family of curves, with the natural non-crossing condition between the two curve systems.

◇ Some degree of regularity (uniform in the short scale distance δ) is needed in order for a description based on random curves to remain meaningful in the scaling limit ($\delta \rightarrow 0$). A generally applicable criterion is developed in ref. [3], where the following result is shown to be implied by a condition, defining Type 1 critical models, which is known to hold for the 2D percolation models. (That the 2D models are of *Type 1* is the result of the Russo [11] and Seymour–Welsh [11, 12] theory, which was recently adapted to a model with spherical-symmetry by K. Alexander [13].)

Theorem 1 (Regularity of connecting curves) *For the two-dimensional site, or bond, critical percolation model, on $\delta\mathbf{Z}^2$ ($\delta < 1$), there is a Hölder continuity exponent, $\alpha > 1/2$ ($= 1/d$) such that all the self-avoiding paths (polygonal with step size delta) along the connected clusters in $[0, 1]^2$ can be simultaneously parameterized by continuous functions $\gamma(t)$, $0 \leq t \leq 1$, satisfying*

$$|\gamma(t_1) - \gamma(t_2)| \leq \kappa(\omega) |t_1 - t_2|^\alpha; \quad \text{for all } 0 \leq t_1 < t_2 \leq 1, \quad (1)$$

where κ is a random variable whose probability distribution does not drift to infinity as $\delta \rightarrow 0$:

$$Prob_\delta(\kappa(\omega) \geq u) \leq g(u) \quad (2)$$

with $g(u) \xrightarrow{u \rightarrow \infty} 0$ uniformly in δ .

This statement provides the required *tightness* of the probability distribution of the Web, which is essential for the existence of a limiting measure, in the scaling limit $\delta \rightarrow 0$. The regularity can alternatively be expressed in terms which are manifestly parametrization independent, as upper bounds on the curves' "tortuosity".

- Lower bounds on the *tortuosity* of the curves are also of some interest, and some such bounds are established, at a similar degree of generality ([3]).
- ◊ The construction leads to an existence result:

Theorem 2 *For $d = 2$ dimensions, and more generally for each dimension in which the critical behavior is Type I, there is a one-parameter family of probability measures (μ_t) on the space of collections of curves (satisfying suitable consistency conditions) each of which has the following properties.*

1. (Independence) *For disjoint closed regions, $A, B \subset R^d$, $W_A(\omega)$ and $W_B(\omega)$ (the restrictions of W to curves in the indicated regions) are independent.*
2. (Euclidean invariance) *The probability measure is invariant under translations and rotations.*
3. (Regularity) *The spanning probabilities of compact rectangular regions are neither 0 nor 1:*

$$R_s := \text{Prob}[\mu] \left(\begin{array}{l} \text{the web configuration } W \text{ includes a path} \\ \text{in } [-s, s]^d \text{ which crosses the cube left} \leftrightarrow \text{right} \end{array} \right) \begin{array}{l} > 0 \\ < 1 \end{array} \quad (3)$$

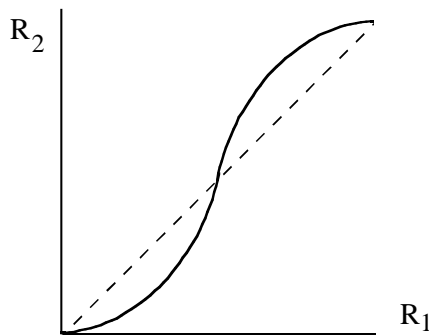


Figure 2: Schematic depiction of the expected shape of the set of values attained jointly by R_1 and R_2 within the one parameter family of random Web measures. The set is expected to form the graph of a map of the interval $[0, 1]$ onto itself with one unstable fixed point, corresponding to the critical case. The slope at the unstable fixed point ($= 4/3?$) yields a critical exponent.

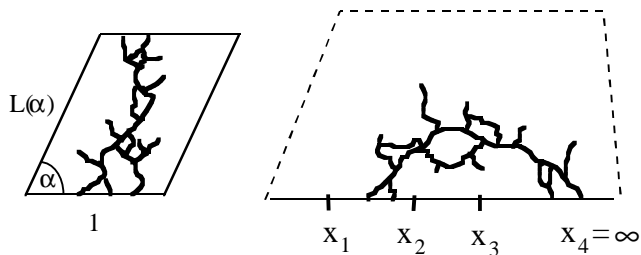
A convenient parametrization for these measures is the crossing probability R_1 . The measures are constructed as continuum limits of sequences of models with suitably adjusted percolation densities. If the standard picture is correct, the density needs be adjusted with the lattice spacing as [14]:

$$p_\delta = p_c + t \delta^{1/\nu} . \quad (4)$$

In order to produce rotation invariant measures the construction is based on the droplet percolation model.

- A direct approach to the continuum theory could be facilitated by a uniqueness statement, which may be presented as an open problem: does the above family contain all the probability distributions of a random Web (the space needs to be defined more carefully than is possible here) having the properties 1) - 3). In general, we are short on arguments proving *uniqueness*. Progress in that area could shed light on a number of issues.
- ◊ The standard renormalization-group picture would require the collection of the joint values of R_1 and R_2 attainable within the above family of Web measures to form a graph of the form depicted in Fig. 2. This observation, which has not been established, shows that the one parameter

- There is accumulating evidence for the conformal invariance of the special scale invariant measure in the above class. The proof can also be reduced to a strong enough uniqueness result [15, 16].



Specific equations have been proposed, and tested, for the crossing probabilities of certain events, such as depicted in Fig. 3, Ref. [4, 1, 8, 9]. These proposals are based on insights derived from other models with known relations to conformal field theory. It should be of interest to understand the equations from within the microscopic picture (for Ising spin systems such a project was started in [17]). Some ideas will be presented in [16] (in preparation). This goal is somewhat reminiscent of the task of deriving the equations of hydrodynamics starting from the microscopic description of gases/fluids, a topic which was discussed in this conference by H-T Yau.

$$F_D(\{\gamma_1, \gamma_2, \gamma_3\}) = \lim_{\delta \rightarrow 0} \text{Prob} \left(\begin{array}{c} \gamma_3 \quad \gamma_2 \\ \text{Diagram of a hexagon with three internal curves meeting at a central point, labeled } \gamma_1, \gamma_2, \gamma_3 \text{ at the bottom, top-left, and top-right respectively.} \\ \gamma_1 \end{array} \right)$$

• There are similarities and relation of the Web with some field theories. The functions depicted in Figure 4 are somewhat reminiscent of vacuum expectation values of certain field operators (of different type for the two cases, the first one discussed in ref. [4]). Intriguing possibilities are suggested by the Fortuin–Kasteleyn random cluster models, which cast Potts models in stochastic geometric terms, and interpolate between the Ising model, whose scaling limit yields the φ_4 field theory, and independent percolation models whose scaling limit is the random Web.

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can appear also through “self-organized” criticality [18]. An examples to which some of the results described here are also relevant is provided by the *minimal spanning tree* process [19, 20, 21].

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